a target. These lead to equations that may be readily implemented in an on-board control algorithm. Using these methods in solving the geometry problem in this work makes it easy to understand the transitions between the different coordinate frames and to implement in a small on-board computer. Straightforward extensions to this work have developed algorithms for single-degree-of-freedom pointing systems (e.g., when the target can be taken to be in the orbital plane) and to invert the problem to derive the target location needed for automatic tracking.

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Low-Earth-Orbit Maintenance: Reboost vs Thrust–Drag Cancellation

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Introduction

WE define the problem of orbit maintenance within an atmosphere as keeping the spacecraft within a specified altitude band about a mean circular orbit. One interesting solution to problem is thrust–drag cancellation,

$$T - D = 0 \tag{1}$$

and thrust vectoring along the velocity vector,

$$\frac{\mathbf{T}}{T} = \frac{\mathbf{v}}{v} \tag{2}$$

resulting in a forced Keplerian trajectory (FKT). Although the control law $T = D$ is quite difficult to achieve physically because of uncertainties in drag modeling (atmospheric density and ballistic coefficient) and thruster designs (on–off), it is typically used to determine the fuel budget required for orbit maintenance.[1] A more practical solution to the orbit maintenance problem is to periodically reboost the spacecraft. Nonetheless, we investigate the fuel optimality of an FKT by considering the totality of extremal arcs. Barring the special case when $T_{\max} = D$, an optimal FKT must necessarily be a singular arc.

It can be shown[2,3] that, when both the thrust magnitude and its direction are control parameters, an FKT is not a Mayer-extremal arc and hence not fuel optimal, i.e.,

$$T^* \neq D, \qquad \frac{\mathbf{T}^*}{T^*} \neq \frac{\mathbf{v}}{v} \tag{3}$$

where the asterisk denotes the optimal values. Unfortunately, this analysis breaks down when Eq. (2) is imposed as a constraint since, although the derivation of the optimal steering is decoupled from that of the thrust magnitude, the converse is not true. Thus, the question remains whether the control law of Eq. (1) is optimal under the steering constraint imposed by Eq. (2): Is $T^* = D$ when $\mathbf{T}/T = \mathbf{v}/v$? Although this question was addressed in Ref. 4 for the special case of a "forced circular orbit," our approach and motivation are quite different in the sense that we seek not only the answer to the more general case of a Keplerian arc but also the ramifications of the extremal solution T^*. To this end, we derive the extremal singular thrust arc T_s^* in state variable feedback form and demonstrate some interesting consequences. In addition, by way of a linear analysis, we show heuristically that the difference in propellant consumption between an FKT and periodic Hohmann transfers is zero (i.e., no greater than the order of the approximations). The following sections elaborate the details of these findings.

Extremal Arcs

The objective of this section is to determine the extremal arcs of a time-free, Mayer-optimal control problem of transferring a spacecraft from some initial manifold to a terminal manifold while minimizing a generic performance index,

$$J = y(r_f, v_f, \gamma_f, m_f) \tag{4}$$

where f denotes the final values and r, v, γ, and m are the variables corresponding to the radial position, speed, flight-path angle, and mass, respectively. The equations of motion for coplanar flight of an endo-atmospheric low-Earth-orbit (LEO) spacecraft are

$$\begin{bmatrix} \dot r \\ \dot v \\ \dot\gamma \\ \dot m \end{bmatrix} = \underbrace{\begin{bmatrix} v \sin\gamma \\ \dfrac{-D}{m} - g\sin\gamma \\ \left(\dfrac{v^2}{r} - g\right)\dfrac{\cos\gamma}{v} \\ 0 \end{bmatrix}}_{a_0} + \underbrace{\begin{bmatrix} 0 \\ \dfrac{1}{m} \\ 0 \\ \sigma \end{bmatrix}}_{a_1} T \tag{5}$$

where the significance of a_0 and a_1 will be apparent later. Here, T is the thrust, D the atmospheric drag, g the gravitational acceleration, and σ the negative inverse of the exhaust speed. These parameters are modeled as

$$D = \tfrac{1}{2}\rho(r)v^2 A C_D, \qquad g = \mu/r^2, \qquad \sigma = -1/g_0 I_{sp} \tag{6}$$

where $\rho(r)$ is a spherically symmetric atmospheric density, A the spacecraft's reference area, C_D the drag coefficient, μ the gravitational constant, I_{sp} the specific impulse, and g_0 the gravitational acceleration at some reference altitude (sea level).

The Pontryagin H-function[5] for this problem is given by

$$H = \lambda_r v \sin\gamma + \lambda_v\left(\frac{T - D}{m} - g\sin\gamma\right) + \lambda_\gamma\left(\frac{v^2}{r} - g\right)\frac{\cos\gamma}{v} + \lambda_m \sigma T \tag{7}$$

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where λ_r, λ_v, λ_γ, and λ_m are the costates corresponding to the state variables r, v, γ, and m, respectively. Application of the maximum principle yields the totality of extremals given by

$$T(\cdot) = \begin{cases} 0 & S < 0 \\ T_s & S \equiv 0 \\ T_{\max} & S > 0 \end{cases} \tag{8}$$

where

$$S = \frac{\lambda_v}{m} + \lambda_m \sigma \tag{9}$$

is the switching function, and T_s is the singular thrust control[6] to be determined. We have assumed that the thrust magnitude is constrained by

$$0 \le T \le T_{\max} \tag{10}$$

Singular Thrust Control

The singular thrust control is obtained by repeated differentiation of the switching function.[6] It is well known[2–4,7] that the result is elegantly expressible by the use of Lie brackets and the first-order singular control is given by[7]

$$T_s = -\frac{\boldsymbol{\lambda}^T[a_0, [a_0, a_1]]}{\boldsymbol{\lambda}^T[a_1, [a_0, a_1]]} \tag{11}$$

where a_0 and a_1 are the vector fields corresponding to the "autonomous" and "controllable" parts, respectively, of the system dynamics [see Eq. (5)]. Although tedious, it is straightforward to show that

$$[a_0, [a_0, a_1]] = \begin{bmatrix} -\dfrac{Ds\gamma}{m^2 v}(2 - \sigma v) + \dfrac{2gc^2\gamma}{mv} \\ \dfrac{2D}{m^2 v^2}\left[\dfrac{D}{m} - gs\gamma(1 - \sigma v)\right] - \dfrac{g^2 c^2 \gamma}{mv^2}\left(1 + \dfrac{v^2}{rg}\right) + \dfrac{D_r s\gamma}{m^2}(1 - \sigma v) - \dfrac{g_r s^2 \gamma}{m} \\ \dfrac{2g^2 c\gamma s\gamma}{mv^3}\left(1 + \dfrac{2v^2}{rg}\right) - \dfrac{Dgc\gamma}{m^2 v^3}\left[(2 - \sigma v)\left(1 + \dfrac{v^2}{rg}\right) + 2\right] \\ 0 \end{bmatrix} \tag{12}$$

$$[a_1, [a_0, a_1]] = \begin{bmatrix} \dfrac{\sigma s\gamma}{m^2} \\ \dfrac{2D}{m^3 v}(1 - 3\sigma v + \sigma^2 v^2) \\ \dfrac{c\gamma}{m^2 v^3}\left[2g + \sigma v\left(g + \dfrac{v^2}{r}\right)\right] \\ 0 \end{bmatrix} \tag{13}$$

where we have employed $s \equiv \sin$ and $c \equiv \cos$ for notational ease. In addition, we have used D_r to denote r partials of D.

The costates along the singular arc may be determined from the first integral and the first and second singular integrals, which are, respectively,[2,3]

$$H = 0 \tag{14a}$$

$$\frac{\partial H}{\partial T} = 0 \tag{14b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial H}{\partial T} = 0 = \boldsymbol{\lambda}^T[a_0, a_1] \tag{14c}$$

These equations result in

$$\lambda_r = \lambda_m \frac{\sigma}{vs\gamma}\left\{D\left[\left(\frac{v^2}{rg} - 1\right)\left(\frac{1 - \sigma v}{2}\right) - 1\right] - \frac{mgs\gamma}{2}\left(1 + \frac{v^2}{rg}\right)\right\} \tag{15a}$$

$$\lambda_v = -\lambda_m \sigma m \tag{15b}$$

$$\lambda_\gamma = \lambda_m \frac{\sigma v m}{2gc\gamma}\left[gs\gamma + \frac{D}{m}(\sigma v - 1)\right] \tag{15c}$$

Since these equations are homogeneous in λ_m, a nonlinear state feedback solution to T_s is possible, and we have

$$T_s = D + \frac{D\{[(p+1)(k^2 - 1) - 2]/s\gamma + s\gamma(6 + 4p - 3k^2 - 3pk^2)\} - mgs^2\gamma - D_r r(1 + p)k^2 s\gamma}{d_w(p^2 + 5p + 3) - s\gamma} \tag{16a}$$

where

$$k^2 = \frac{v^2}{rg}, \qquad d_w = \frac{D}{mg}, \qquad p = -\sigma v \tag{16b}$$

k is the Kepler number, d_w the drag-to-weight ratio, and p a propulsion parameter; each of these is positive. It is clear that

$$T_s - D \neq 0 \tag{17}$$

under any circumstance. It is noteworthy that $\gamma = 0$ (a necessary condition for a forced circular orbit) does not lie on the singular surface since, as $\gamma \to 0$, $T_s \to \infty$. This is a result of the preimposed steering law $\boldsymbol{T}/T = \boldsymbol{v}/v$. In fact, this steering law alters the structure of the singular arc. This is because since the order of the singular arc (half the number of repeated differentiations of the switching function) for the exoatmospheric case is two,[6] we must have $[a_1, [a_0, a_1]] = 0$ [see Eq. (11)] whenever $D = 0$. This is equivalent to the condition[3]

$$T_s \to \infty \qquad \text{as} \qquad D \to 0 \tag{18}$$

However, from Eqs. (16), it is clear that

$$T_s \to mgs\gamma \qquad \text{as} \qquad D \to 0 \tag{19}$$

is a finite quantity. Thus, the steering law has created an "artificial" singular arc.

Fuel Efficiency of Orbit Raising

From the previous sections, an FKT is not an extremal arc of the Mayer-optimal control problem, and hence it cannot be a subarc of an optimal trajectory. Accordingly, for orbit maintenance, periodic boosting must provide fuel-efficient trajectories. Although this result indicates what is not optimal, it does not however tell us how the reboost maneuvers must be performed or whether the thrusting is singular or maximum. For the purpose of exploring the utility of this result, we wish to compare analytically the fuel required for an FKT with a periodic Hohmann transfer by way of a linear analysis.

Suppose the orbit of a spacecraft contracts (due to atmospheric drag) from its initial circular orbit of radius r down to $r - \Delta r$ in time $t_D \gg \tau = 2\pi r/v$, where τ is a first-order approximation to the orbital period. To perform a Hohmann transfer from $r_1 = r - \Delta r$ to $r_2 = r$, the required Δv is given by[1]

$$\Delta v = \sqrt{\frac{\mu}{r_1}}\left[\sqrt{\frac{2(r_2/r_1)}{1 + r_2/r_1}}\left(1 - \frac{r_1}{r_2}\right) + \sqrt{\frac{r_1}{r_2}} - 1\right] \cong v\frac{\Delta r}{2r} \tag{20}$$

where we have assumed that $\Delta r/r \ll 1$ and $v = \sqrt{\mu/r_1}$. Thus, the Δv budget per unit time for a periodic Hohmann transfer can be approximated by

$$\Delta v_{\text{Hohmann}} \cong \Delta v/t_D = v\,\Delta r/2rt_D \tag{21}$$

The first-order change in orbital radius per orbit is given by[1]

$$\delta r = 2\pi C_D A\rho r^2/m \tag{22}$$

Since $\Delta r = \delta r\, t_D/\tau$, Eq. (21) reduces to

$$\Delta v_{\text{Hohmann}} = v\,\delta r/2r\tau = D/m \tag{23}$$

The first-order change in orbital speed per orbit is given by[1]

$$\delta v = \pi C_D A\rho r v/m \tag{24}$$

Thus, the Δv budget per unit time for an FKT can be approximated by

$$\Delta v_{\text{FKT}} \cong \delta v/\tau = D/m = \Delta v_{\text{Hohmann}} \tag{25}$$

Conclusions

An FKT is not a fuel-optimal maneuver since it is not a singular arc. For LEO maintenance, any savings in propellant accomplished by a periodic Hohmann maneuver is a higher order effect. Since the optimal maneuver is unknown, we have at least two possibilities: (1) the propellant consumed by an FKT is close to the optimal if a periodic Hohmann maneuver is also close to the optimal or (2) since the propellant consumed by a Hohmann-type maneuver is close to the nonoptimal FKT, the periodic optimal maneuver is quite different from the Hohmann maneuver, possibly consisting of singular subarcs. One way to resolve this question is to develop a minimum-fuel, finite-burn maneuver by using periodic optimal control theory and compare its performance to that of an FKT.

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Near-Optimal Three-Dimensional Rotational Maneuvers of Spacecraft Using Manipulator Arms

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Introduction

THIS Note presents a framework for obtaining near-optimal three-dimensional rotational maneuvers of spacecraft possessing multiple interconnected manipulator arms. In the absence of external torques, the angular momentum of the spacecraft is conserved and this imposes a constraint on its motion. The nonholonomic nature of this constraint allows for the design of open-loop control profiles for positioning the spacecraft attitude as well as the joint angles, as shown by Reyhanoglu and McClamroch[1] on a planar example. The method was extended to three-dimensional maneuvers by Mukherjee and Zurowski.[2]

Two types of maneuvers are presented in this Note. The first one minimizes joint accelerations with specified final time and the second one minimizes the maneuver time with specified bounds on the

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